1 What Is Infinity?

Infinity is a Loch Ness monster, capturing the imagination with its awe-inspiring size but elusive nature. Infinity is a dream, a vast fantasy world of endless time and space. Infinity is a dark forest with unexpected creatures, tangled thickets, and sudden rays of light breaking through. Infinity is a loop that springs open to reveal an endless spiral.

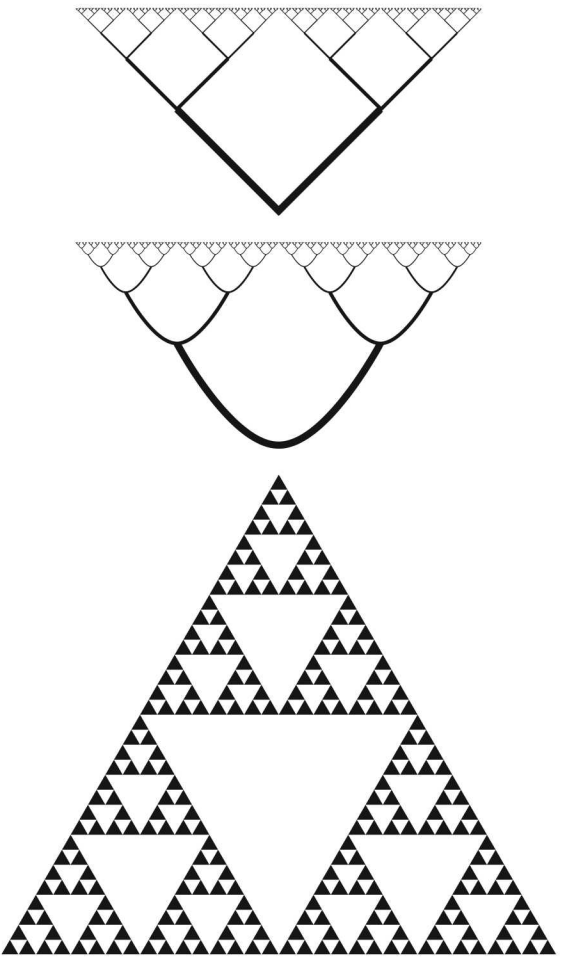
Our lives are finite, our brains are finite, our world is finite, but still we get glimpses of infinity around us. I grew up in a house with a fireplace and chimney in the middle, with all the rooms connected in a circle around it. This meant that my sister and I could chase each other round and round in circles forever, and it felt as if we had an infinite house. Loops make infinitely long journeys possible in a finite space, and they are used for racetracks and particle colliders, not just children chasing each other.

Later my mother taught me how to program on a Spectrum computer. I still smile involuntarily when I think about my favorite program:

```
10 PRINT "HELLO"
20 GOTO 10
```

This makes an endless loop – an abstract one rather than a physical one. I would hit `RUN` and feel delirious excitement at watching `HELLO` scroll down the screen, knowing it would keep going *forever* unless I stopped it. I was the kind of child who was not easily bored, so I could do this every day without ever feeling the urge to write more useful programs. Unfortunately this meant my programming skills never really developed; infinite patience has strange rewards.

The abstract loop of my tiny but vast program is made by the program going back on itself, and self-reference gives us other glimpses of infinity. Fractals are shapes built from copies of themselves, so if you zoom in on them they keep looking the same. For this to work, the detail has to keep going on "forever," whatever that means – certainly beyond what we can draw and beyond what our eye can see. Here are the first few stages of some fractal trees and the famous Sierpinski triangle.



If you point two mirrors at each other, you see not just your reflection, but the reflection of your reflection, and so on for as long as the angle of the mirrors permits. The reflections inside the reflections get smaller and smaller as they go on, and in theory they could go on "forever" like the fractals.

We get glimpses of infinity from loops and self-reference, but also from things getting smaller and smaller like the reflections in the mirror. Children might try to make their piece of cake last forever by only ever eating half of what's left. Or perhaps you're sharing cake, and everyone is too polite to have the last bite so they just keep taking half of whatever's left. I'm told that this has a name in Japanese: *enryo no katamari*, the last piece of food that everyone is too polite to eat.

We don't know if the universe is infinite, but I like staring up at a church spire and tricking myself into thinking that the sides are parallel and it's actually an infinite tower soaring up into the sky to infinity. Our lives are finite, but fictional and mythological tales of immortality appear through the ages and across cultures.

So much for glimpses of infinity, like ripples in the waters of Loch Ness that may or may not have been caused by a giant, ancient, mysterious monster. What is this monster we call infinity? What do we mean when we throw around the innocuous-sounding word "forever"? In our modern impatient world, people are prone to using it rather hyperbolically, exclaiming "I've been on hold *forever*!" after two minutes, or "This web page is taking *forever* to load!" if it takes more than about three seconds. I'm told by Basque writer Amaia Gabantxo that in Basque the word for eleven is *hamaika*, but it also means infinity. This is borne out by another friend of mine who did an audit of homemade jam in the cupboard and declared there were "four jars from 2013, ten from 2014, and lots from 2015." Apparently more than ten might as well be infinity. My research field is higher-dimensional category theory, and there "higher" usually means three dimensions or more, including infinity – everything from just three to infinity turns out to be about the same.

The way we think about infinity in our normal lives might be dreamy and exciting, but it dissipates under close examination, like the end of a rainbow, which will never be there if you try to go looking for it. It causes paradoxes and contradictions, impassable ravines and murky traps. It doesn't stand up to the tests of rigorous logic, as we'll see.

One of the roles of mathematics is to explain phenomena in the world around us, especially phenomena that crop up in many different places. If a similar idea relates to many different situations, mathematics swoops in and tries to find an overarching theory that unifies those situations and enables us to better understand the things they have in common. Infinity is one of those ideas. It pops up all over the place as an idea that we can dream about, and seems similar to other ideas that *can* be unified by mathematics: the ideas of length, size, quantity. So why is it so difficult to extend those easy mathematical concepts to include infinity? This is what this book is about: why it is difficult, how it eventually can be done, and what we see along the way.

Infinity Instincts

Infinity is easy to think about but hard to pin down. Small children can quickly latch on to the idea of infinity, but it took mathematicians thousands of years to work out how to explain infinity in full-technicolor logical glory. Here are some things we might think about infinity. Children come up with these thoughts about infinity quite often, all by themselves:

Infinity goes on forever.
Infinity is bigger than the biggest number.
Infinity is bigger than anything we can think of.
If you add one to infinity it's still infinity.
If you add infinity to infinity it's still infinity.
If you multiply infinity by infinity it's still infinity.

Children can get very excited when the notion of infinity first dawns on them. They learn to count up to ten, and then twenty, and then they learn about a hundred, a thousand, a million, a billion. If you ask a small child what the biggest possible number is, they may well say "a billion," but then you can ask them about a billion and one and watch their eyes widen.

It's not very hard to convince them that no matter what number they think of, you could always add one and get a bigger number. This gives the idea that there is *no biggest number*. Numbers go on forever! But then, how many numbers are there in total? The idea of infinity starts to emerge.

Perhaps some children first hear of infinity because they watch the *Toy Story* films and they hear Buzz Lightyear saying, "To infinity . . . and beyond!," which does sound very exciting. When I was a child *Toy Story* hadn't been made yet, but I had inklings of infinity from the loops I described earlier, the physical loops in our house and the abstract loops in my favorite computer program.

Once children start thinking about infinity, they can easily come up with questions about it that are teasingly difficult to answer. What is infinity? Is it a number? Is it a place? If it's not a place, how can we go there and beyond?

If children hear about infinity at school, the questions just start proliferating. Is one divided by zero infinity? Is one divided by infinity zero? If infinity plus one is infinity, what happens when you subtract infinity?

When children ask innocuous math questions that seem impossible to answer, this can be intimidating for adults, who feel that they are supposed to have all the answers. But as math educator and innovator Christopher Danielson says, an important aspect of learning is being able to ask new questions; this is more important than being able to state new facts. In math, there are always more questions. Even people who did quite well at math, even people who did math at university, even

research mathematicians have more questions about infinity than can be answered.

Infinity Weirdness

Here are some of my favorite mind-boggling conundrums about infinity that we are going to explore.

* If you have an infinite hotel and it's full, you can still fit another guest in by moving everyone up one room.
* If there were a lottery with infinitely many balls, what would be your chances of winning?
* Some infinities are bigger than others!
* Infinite pairs of socks are somehow more infinite than infinite pairs of shoes.
* If I were immortal, I could procrastinate forever.
* If you are traveling from A to B, you first have to cover half the distance, then half the remaining distance, and then half the remaining distance, and so on. There will always be half the remaining distance left, so you'll never get there. Or will you?
* The recurring decimal $0.\dot{9}$ *is exactly equal to* 1.
* Does a circle have infinitely many sides?
* Why do people who do quite well at math often get stuck at calculus? Yes, this is a question about infinity too.

Infinity can appeal to people of all ages, of all levels of expertise, in different ways. This book is going to be a journey to infinity – and beyond. Because there really is something beyond infinity, if you think about it hard enough and in the right way. Just like there are always more questions to ask and more things to understand. Infinity is not a physical place, so this is not a physical journey. You can go with me on this journey sitting right where you are, because the journey is an *abstract* journey.

It's a journey into the deep, tangled, mysterious, murky, endless world of ideas.

Why?

Why do we go on this journey? Like with physical journeys, there are many reasons to go on an abstract journey. Everyone has their own reasons for traveling. Maybe there's something particular you want to do at your destination. Maybe there's a really good view from the top. Maybe there's beautiful scenery along the way. Maybe you enjoy the physical exertion of walking or climbing, or the exhilaration of riding a bicycle or driving a fast car, or the serenity of sitting on a train watching the countryside rush past. (My experience of trains involves more delays and irate commuters than serenity, but let's ignore that for a second.) Maybe you like going into the unknown. Maybe you like wandering around and losing yourself completely in a city. Maybe you have the travel bug and want to see as much of our incredible earth as possible, simply because there is so much to see.

All of these reasons have their counterparts in the abstract world. A journey where you have something particular to do (like commuting to work) corresponds to having a particular problem you want to solve or a particular application in mind. This type of abstract journey is less about the sense of discovery and more about getting something done. Going up high for a good view is like the abstract investigations we do to gain new perspectives on things we've already seen up close. The beautiful scenery along the way is the weird and wonderful ideas and scenarios that come up as we investigate. And yes, there is exhilaration in the mental exertion, and excitement in thinking about ideas that seem incomprehensible and then slowly clear up, like fog clearing to reveal the ocean glistening to the horizon. I don't have the travel bug for the physical

world as much as some people do, but I do have the curiosity bug, the counterpart for the abstract world. I'm fairly calm and resigned about there being a lot of the world I haven't seen, but I am insatiable when it comes to ideas I don't understand. I always want to explore them. As soon as I catch a glimpse of something I don't understand, I am driven to plunge into it. I like losing myself completely in a city, and I like losing myself in ideas as well. As much as I am driven to understand things, I am also happy to acknowledge the things we humans *cannot* understand. In fact, I positively revel in it. It means that there is always more out there, which is a glorious thing. Wouldn't it be a bit sad if you could finally say yes, that's it, I've been to every restaurant in London now? But of course that's impossible. There will *always* be another restaurant you haven't tried, and there will *always* be things we don't understand.

In a strange way this book isn't about infinity at all. It's about the excitement of a journey into the abstract unknown. Jules Verne's *Journey to the Center of the Earth* wasn't really about the center of the earth, but about the excitement of an incredible journey. This is a book about how abstract thinking works and what it does for us. It's about how it helps us pin down what we really mean when we start having an interesting idea. It doesn't necessarily explain the whole idea: mathematics doesn't explain everything about infinity. But it does help us become clear about what we can and can't do with infinity.

This first part of the book, then, is a journey toward understanding what infinity is. If you ask a small child what they think infinity is, they might say something like "It's bigger than any number you can think of." This is true, but it still doesn't tell us what infinity *is*, any more than saying "Yao Ming is taller than anyone you've ever met" tells you who Yao Ming is.

In the second part of the book we'll take a tour of the world equipped with our new ideas about infinity and see where this elusive creature has been lurking all along. It's in mirrors that

we point at each other, racetracks that we run around, every journey we take, and in every changing situation in this continuously changing world of ours. Understanding infinity is the basis of the field of calculus, which is inextricably embedded in almost every aspect of modern life.

Of course, it is possible to enjoy all those aspects of modern life without having the slightest understanding of calculus, which is why I don't emphasize these applications as the main reason for writing about infinity. Mathematics suffers a strange burden of being required to be useful. This is not a burden placed on poetry or music or football. If you ask me what all this is useful for, I could answer by saying it helps us generate electricity; make phone calls; build bridges, roads, and airplanes; provide water to cities; develop medicines; save lives. But this doesn't mean it is useful for *you* to think about it, only that it is useful to you that *somebody else* thought about it. It is not why I think about it, and not why I most want to tell this story.

You can get by perfectly well in life without understanding anything more about infinity than you did when you were five years old. But for me the usefulness of mathematics isn't about whether you need it to "get by" in life or not. It's about how mathematical thinking and mathematical investigation sheds light on our thought processes. It's about taking a step back from something to get a better overview. Flying higher up in the sky enables us to travel farther and faster.

Let's go.